NONLINEAR BAND-PASS FILTERING USING THE TV TRANSFORM

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ABSTRACT

A distinct family of nonlinear filters is presented. It is based on a new formalism, defining a nonlinear transform based on the TV-functional. Scales in this sense are related to the size of the object and its contrast. Edges are very well preserved and selected scales of the object can be either selected, removed or enhanced. We compare the behavior of the filter to other filters based on Fourier and wavelets transforms and present its unique qualities.

Index Terms— Total variation, TV transform, spectral TV, nonlinear filtering.

1. INTRODUCTION

Total-variation (TV) is the archetypical edge-preserving functional. It is extensively used in image processing in the past two decades, see e.g. [1–6]. Since its introduction in [1] in the context of image processing many studies have been devoted to its analysis and interpretation, e.g. [3, 4, 7, 8]. In [9, 10] a non-conventional way of defining a transform through a nonlinear partial-differential-equation (PDE) was suggested.

Filtering based on transforms and spectral analysis has been used extensively in the analysis and processing of signals modeled as stationary random processes (see e.g. [11, 12]). For more complex non-stationary signals, such as images and speech, harmonic analysis methods were developed in the form of wavelets [13–15], spectral graph theory [16] and diffusion maps [17]. In this work we further develop the applicability of the TV transform to filtering in general and band-pass filtering in particular, comparing it to more classical filtering techniques.

In [18] Steidl et al have shown the close relations, and equivalence in a 1D discrete setting, of the Haar wavelets to both TV regularization [1] and TV flow [19]. This was later developed in 2D for a more restricted setting [20]. In this work we show the conceptual resemblance of the TV-transform to Haar-wavelet filtering, yet we observe the limitations of the wavelet technique and show the superior results obtained with the new TV-based method to obtain scale separation and high quality filtering.

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This work relies on the established theory of the TV flow proposed by Andreu et al in [19] and further developed in [21–23] and the references therein.

1.1. TV functional and TV flow

The total variation functional is:

$$J(u) = \int_\Omega |Du|, \tag{1}$$

where Du denotes the distributional gradient of u. The total variation scale-space, known as total-variation flow [19], is formally written as:

$$\begin{array}{ll} \frac{\partial u}{\partial t} = \text{div}\left(\frac{Du}{|Du|}\right), & \text{in } (0,\infty)\times\Omega \\ u(0;x) = f(x), & \text{in } x \in \Omega, \end{array} \tag{2}$$

where Ω is the image domain (a bounded set in $\mathcal{R}^N$ with Lipschitz continuous boundary $\partial\Omega$) and $f(x)$ is the input image, Neumann boundary conditions are assumed.

2. THE TV TRANSFORM

In [9, 10] a nonlinear transform was defined, based on TV. It was argued that many properties and intuitions, related to classical transforms, remain valid also in this nonlinear setting. To derive a meaningful nonlinear transform the following requirements were set:

1. Certain spatial structures (“atoms” of the functional) are transformed to impulses in the transform domain.
2. The inverse transform can recover any spatial signal (within a defined space) from the transform domain components.
3. Filtering can be applied in the transform domain by simple attenuation or amplification of the transform components.
4. The spectrum represents significant scales of the image.

The TV transform is defined by:

$$\phi(t;x) = u_{tt}(t;x)t, \tag{3}$$

where u_{tt} is the second time derivative of the solution $u(t;x)$ of the TV-flow equation (2). The inverse transform is:

$$f(x) = \int_0^\infty \phi(t;x)dt + \bar{f}, \tag{4}$$

where $\bar{f} = \frac{1}{\Omega}\int_\Omega f(x)dx$ is the mean value of the initial condition. Finally, the spectrum $S(t)$ corresponds to the L^1 amplitude of each scale:

$$S(t) = \|\phi(t;x)\|_{L^1} = \int_\Omega |\phi(t;x)|dx. \tag{5}$$

Two significant results were shown in [10] for this transform:

1. **Atoms as eigenfunctions:** Let $f(x)$ be a function which admits the nonlinear eigenvalue problem: $pf = \alpha f$, where $p \in \partial_f J$ is the subdifferntial of $J(f)$, and $\alpha \in \mathbb{R}^+$. Then the transform yields a single impulse, multiplied by $f(x)$, at some time $t = t_d$ and is zero for all other t.
2. **Relations to TV-flow:** The TV flow solution $u(t_1;x)$ is a specific low-pass filter in this framework.

The first result relates to nonlinear spectral theory [24], which has attracted increasing interest lately, see e.g. [25]. The implication is that for eigenfunctions of the functional we construct filters with perfect reconstruction or suppression ("ideal filtering"). This leads to very high quality separation of scales in the image.

The second result shows that the framework is a generalization of standard TV filters and that many other new filters related to the functional can be designed.

The TV-transform decomposition can be seen as a generalization and extension of earlier studies concerning image decomposition methods, such as [4, 26, 27].

3. TV BAND-PASS FILTERING

Filtering is defined by a non-negative amplification function $H(t)$:

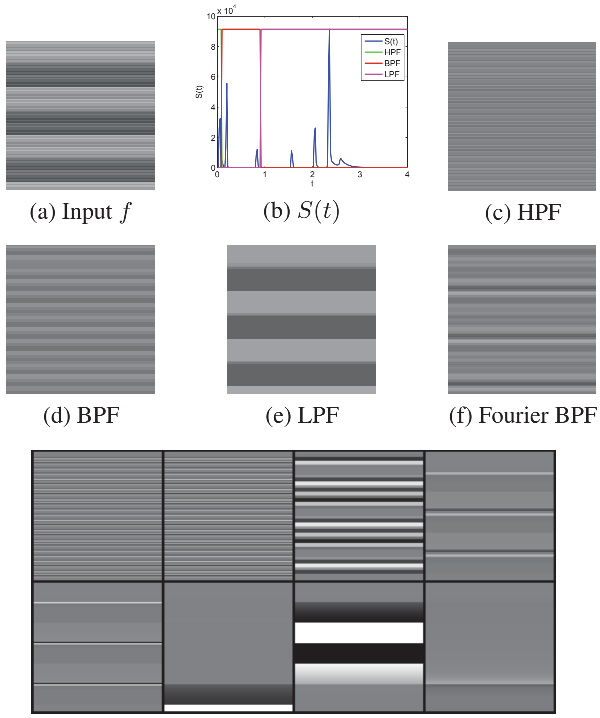
$$\phi_H(t;x) = \phi(t;x)H(t), \tag{6}$$

The filtered response in the spatial domain is then the following reconstruction procedure:

$$f_H(x) = \int_0^\infty \phi_H(t;x)dt + H(\infty)\bar{f}. \tag{7}$$

The scale size is proportional to the time parameter t. As t grows, $\phi(t)$ contain coarser scale features. For the TV transform, scale is a combination of contrast and spatial size. In the case of a disk of radius r and height h, it can be shown that t is linearly proportional to the radius multiplied by the height: rh. Let us consider two disks i, j, which are well separated, each with radius and height $r_{i/j}$, $h_{i/j}$, respectively. When the following condition holds:

$$r_i h_i = r_j h_j,$$



(a) Input f (b) $S(t)$ (c) HPF
(d) BPF (e) LPF (f) Fourier BPF
(g) $\phi(t;x)$ of some scales

Fig. 1. Simple example of separating mixed scales of stripes using TV-based filtering.

$j \neq i$, the disks are within the same $\phi(t;x)$ component. Therefore, under this transform, they are considered to be in the same scale. Other variations, for instance ones which consider only spatial size and are contrast invariant, are being investigated now.

A band of scales is therefore all scales within some range $[t_1, t_2]$, $0 \le t_1 < t_2 \le \infty$. In the case $t_1 = 0$ we have all scales from the finest one, this reduces to a high-pass-filter. In the case of $t_2 = \infty$ we get all coarse scales (where the ∞ scale stands for the mean value of the signal) and thus the filter reduces to a low-pass-filter.

In the general case, a band-pass-filter is strongly attenuating non-desired scales and maintains well the desired components. For simplification we use the ideal band-pass-filter for which each scale is either omitted completely or maintained fully. A band-pass filter is defined as

$$H_{BPF,t_1,t_2}(t) = \begin{cases} 0, & 0 \le t < t_1 \\ 1, & t_1 \le t < t_2 \\ 0, & t_2 \le t \le \infty \end{cases} \tag{8}$$

A band-stop filter can be defined as the complement (any 0 becomes 1 and vice versa).

Nonlinear band-pass filters have not been employed frequently in image analysis. However, they can reveal important characteristic of the signal. They are able to extract significant cues for classification which may be hidden under the

(a) Input f

(b) Hidden pattern

(c) Fourier

(d) Wavelets, DB-4

(e) Wavelets, Haar

(f) Features, Haar

(g) Proposed

(h) Features, proposed

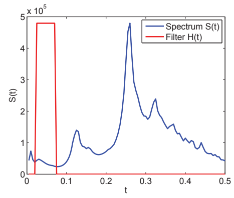
Fig. 2. Extracting a well hidden pattern using different methods.

clutter of low and large scales and can serve to design new types of image descriptors for medical and other purposes. In the examples below we show some uses of band-pass filtering, compare the proposed method to more classical ways and discuss the method's unique qualities.

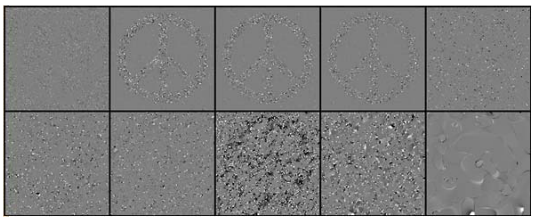
4. EXAMPLES AND COMPARISON

In Fig. 1 we show a simple case of an input image with stripes of several scales. We try to separate those scales into 3 components using TV-based low-pass-filter (LPF), band-pass-filter (BPF) and high-pass-filter (HPF). We show the Fourier band-pass analogue, which has similar qualities for such repetitive signal. These transforms deviate considerably for non-periodic signals.

In Fig. 2 we attempt to recover a well hidden pattern by various methods. We use Fourier and two wavelet methods, Daubechies 4 and Haar. It is interesting to see that Daubechies 4 cannot recover the pattern (we tried various bands) whereas Haar is quite successful. Nevertheless, the extraction is not as good as the proposed method. The features taken are smooth-



(a) Spectrum $S(t)$



(b) Instances of $\phi(t;x)$

Fig. 3. Extracting pattern - spectrum and some ϕ examples of the proposed method.

ing of the absolute value of the band. In Fig. 3 the spectrum and some ϕ instances are shown (note on the top middle the ϕ's containing the pattern).

In Fig. 4 large scale separation is shown. It is clear that the proposed method can best isolate the different scales at the best quality.

In Fig. 5 histology image (depicting Metaplasia) is shown and its separation to different scales. Our assessment is that different features and information can be analyzed more precisely in this manner. This should be verified experimentally in future studies.

In Fig. 6 a leaf and its decomposition is depicted. At the bottom right, a band-stop filter result is shown with both coarse and fine scales, but with the middle scales extracted. This may be useful when the middle scales obstruct or when one wants to put the fine scale data within context.

5. CONCLUSION

The paper suggests the use of nonlinear band-pass filtering, based on the TV-transform, for image analysis and representation. This allows significant information and signal cues, often hidden under high contrast features, to be revealed.

The suggested method can separate and isolate image features of different scales, in a well defined manner, without intrinsic process parameters (no tuning needed). We have compared the method to linear (Fourier) band-pass filtering and wavelet-based methods. Although there is some common qualities of the TV-based method and the Haar-wavelets approach, the proposed approach outperforms all methods in terms of robustness and high quality separation. The method can serve as a building block to extract features for machine-learning algorithms and can serve in the design of new types of image descriptors.

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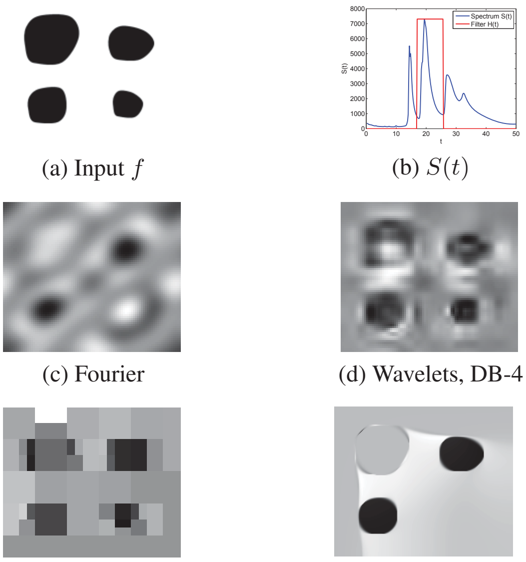
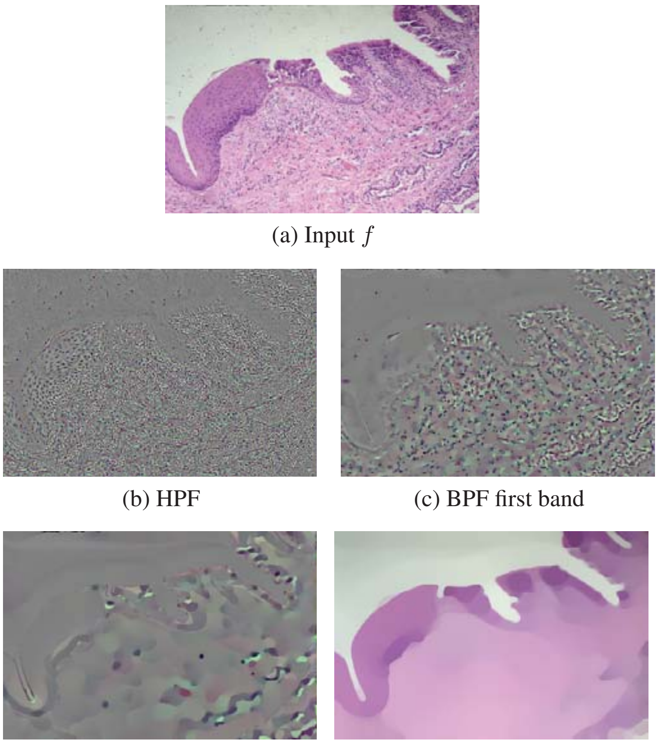


(a) Input f (b) $S(t)$

(c) Fourier (d) Wavelets, DB-4

(e) Wavelets, Haar (f) Proposed

Fig. 4. Isolating large scale features using different methods.

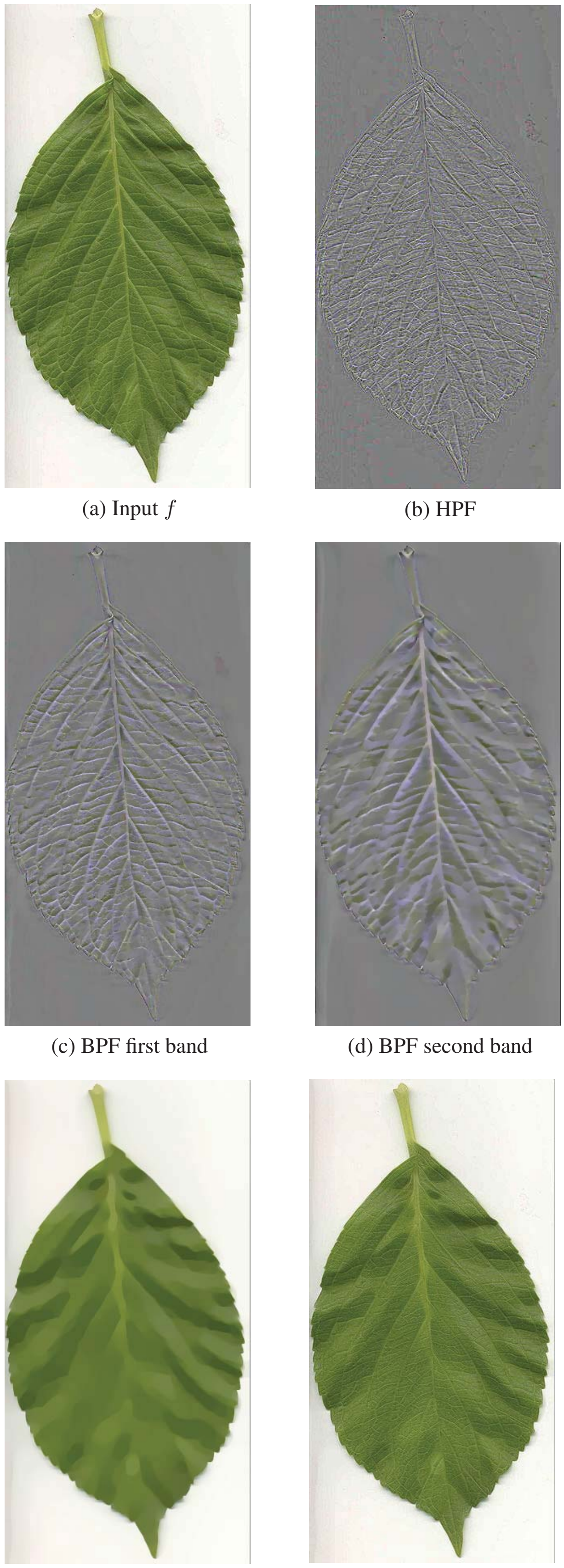


(a) Input f

(b) HPF (c) BPF first band

(d) BPF second band (e) LPF

Fig. 5. Extracting different feature scales of a histology data.



(a) Input f (b) HPF

(c) BPF first band (d) BPF second band

(e) LPF (f) Band-stop filter

Fig. 6. A leaf and its different scales.